

$DW(2n, q)$, $n \geq 3$, has no ovoid: A single proof

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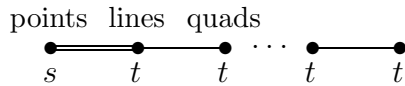
Abstract

An ovoid of a dual polar space Δ is a point set meeting every line of Δ in exactly one point. For the symplectic dual polar space $DW(6, q)$, Cooperstein and Pasini [2] have recently proved no ovoid exists if q is odd. Earlier, Shult has proved the same for even q (cf. [3, 2.8]). In this paper, we prove the non-existence of ovoids in $DW(6, q)$ independently from the parity of q .

MSC 2000: 51A15, 51A50, **Key words:** dual polar spaces, hyperplanes, ovoids.

1 Introduction

Let Δ be a finite dual polar space of finite rank $n \geq 3$, i.e. it is the geometry dual of a polar space Π . The points of Δ are the $(n-1)$ -dimensional singular subspaces of Π , the lines of Δ are the $(n-2)$ -dimensional singular subspaces of Π and, more generally, the elements of type i of Δ , $i = 1, \dots, n$, are the $(n-i)$ -dimensional singular subspaces of Π . Thus Δ belongs to the diagram



An ovoid of a dual polar space is a point set that meets each line in exactly one point. It is an outstanding conjecture that in finite classical dual polar spaces of rank at least three no ovoid exists. By Pasini and Shpectorov [3], the complement of an ovoid of a dual polar space of rank 3 cannot be flag-transitive. Recently, Cooperstein and Pasini [2] have proved that the finite symplectic dual polar space $DW(6, q)$ for q odd has no ovoid. Together with a similar non-existence result for q even by Shult to be found in [3, 2.8], $DW(6, q)$ has no ovoid.

Since we restrict the considerations to finite dual polar spaces Δ , the polar space Π dual of Δ is classical by Tits' classification of polar spaces. Suppose O is an ovoid of Δ . The ovoid O intersects each quad of Δ in an ovoid of a generalized quadrangle. The quads are either grids if $t = 1$ or classical generalized quadrangles of order (s, t) . Since we assume they admit ovoids and lines aren't short, i.e. $t \geq 2$, Δ is the dual of the symplectic polar space $W(6, q)$, the orthogonal polar space $O^-(8, q)$ or possibly the hermitian polar space $H(7, q^2)$, $q \geq 3$, with the quads being generalized quadrangles $O(4, q)$, $H(4, q^2)$ or the dual of $H(5, q^2)$, respectively (for the classical generalized quadrangles admitting ovoids, see Payne and Thas [4]). Note that the existence of an ovoid of the dual of the hermitian generalized quadrangle $H(5, q^2)$, $q \geq 3$, i.e. a spread of $H(5, q^2)$, is an open problem.

In this paper, we simplify the proof of [2] and generalize it such that we can apply it to any dual polar space of rank 3 hypothetically admitting ovoids. For the classical dual polar spaces, the counting leads to a contradiction only for the symplectic dual polar space $DW(6, q)$. However, it is independent from the parity of q , whence our new prove comprises the results of Shult ([3, 2.8]) and Cooperstein and Pasini [2].

Theorem 1 *If Δ has an ovoid, then $s^2 - s - t^2 - t \geq 0$. In particular, $DW(6, q)$ has no ovoid.*

Theorem 1 generalizes to symplectic dual polar spaces $DW(2n, q)$ of finite rank $n \geq 4$ since the point-line residue of each element of type 4 of $DW(2n, q)$ is a dual polar space $DW(6, q)$ which would intersect an ovoid of $DW(2n, q)$ in an ovoid of $DW(6, q)$ in contradiction to Theorem 1.

Corollary 2 *A finite symplectic dual polar space $DW(2n, q)$, $n \geq 3$, has no ovoid.* \square

2 Proof of the Theorem

We recall the notation and introduce some more terminology. For an element x of a geometry G of diameter d , for $i = 1, \dots, d$, $G_i(x)$ denotes the set of points of G at distance i from x in the collinearity graph of G .

Before starting the proof, we introduce the projection π_E of the point set of a dual polar space onto the point set of an arbitrary element E of type at least 2. Since dual polar spaces of rank n are near $2n$ -gons (cf. Cameron [1]), for any point p and any element E of the dual polar space not on p , there is a unique point $\pi_E(p)$ in E nearest p . In setting $\pi_E(x) = x$ for all points x in

E , the so-defined mapping π_E maps two collinear points of Δ either onto the same point of E or onto two collinear points of E . Thus, if Δ is a dual polar space of rank 3, the following proposition which mainly serves as reference for Proposition 5, is immediate.

Proposition 3 *If ω is a quad of Δ , then a line l disjoint from ω is mapped onto the line $\pi_\omega(l)$ of ω such that for each point p on l , there is a unique point on $\pi_\omega(l)$ collinear with p . \square*

Let us now turn to the proof of Theorem 1. We suppose Δ is a finite dual polar space of rank 3 and the generalized quadrangle consisting of the points and lines of a quad of Δ has order (s, t) with $t \geq 2$. Moreover, since we assume Δ has an ovoid O and the quads are classical generalized quadrangles, it follows $s \geq t$ (cf. Payne and Thas [4]).

Let ∞ be a point of O . Then the set H of points of Δ at non-maximal distance from ∞ is the so-called *singular hyperplane with deepest point* ∞ , whence $H = \{\infty\} \cup \Delta_1(\infty) \cup \Delta_2(\infty)$. Denote the affine dual polar space consisting of the elements of Δ not contained in H by $\Gamma := \Delta - H$, i.e. the point set of Γ is $\Delta_3(\infty)$. We call the lines of Δ not contained in the singular hyperplane H the *affine lines* of Γ . Moreover, we set $\Omega := O \cap \Gamma$.

Since both O and H are hyperplanes of Δ , a line l of Δ not contained in H has exactly one point $l^O = l \cap O$ of the ovoid O and one point $l^\infty = l \cap H$ of H . The main idea of Cooperstein and Pasini [2] is to count pairs (l, m) of concurrent affine lines of the affine dual polar space $\Gamma = \Delta - H$ such that the unique point l^∞ of the line l of Δ in H lies in O , i.e. $l^\infty = l^O$, and the unique point m^∞ of the line m of Δ in H does not belong to O , i.e. $m^\infty \neq m^O$. Using Cauchy's inequality, their final conclusion for $\Delta = DW(6, q)$, q odd, is $2q \leq 0$ proving no ovoid exists.

We follow most of the proof of Cooperstein and Pasini [2] and most of their notation. The only modification is the method to prove Proposition 5 below. It allows us to apply the final argument of [2] to arbitrary finite dual polar spaces. In particular, it includes $DW(6, q)$ for q even.

Proposition 4 *It holds $|\Gamma| = s^3 t^3$, $|O| = (st + 1)(st^2 + 1)$, and $|\Omega| = st^3(s - 1)$.*

Proof. Since $|\Delta| = (t + 1)(st + 1)(st^2 + 1)$, $|\Delta_1(\infty)| = (t^2 + t + 1)s$ and $|\Delta_2(\infty)| = |\Delta_1(\infty)|(t^2 + t)s/(t + 1) = (t^2 + t + 1)st$, it follows

$$|\Gamma| = |\Delta| - |\Delta_2(\infty)| - |\Delta_1(\infty)| - 1 = s^3 t^3 .$$

We determine $|O|$ by

$$|O| = \frac{\# \text{ lines}}{\# \text{ lines per point}} = \frac{(st+1)(st^2+1)(t^2+t+1)}{t^2+t+1} = (st+1)(st^2+1) .$$

Since each quad on ∞ has st points of $O - \{\infty\}$ and none of the points of O at distance two from ∞ belongs to two quads on ∞ , it follows

$$|\Omega| = |O| - (t^2 + t + 1)st - 1 = st^3(s - 1) . \quad \square$$

Following Cooperstein and Pasini [2], for a point $p \in \mathcal{G} := \Gamma - \Omega$ let

$$\mu_p := |\Gamma_1(p) \cap \Omega|$$

be the number of ovoid points collinear with p not belonging to the singular hyperplane H . Since there are $t^2 + t + 1$ lines on p all meeting O in exactly one point, $t^2 + t + 1 - \mu_p$ lines on p meet O in points of $O - \Omega$. Then the number of pairs of concurrent lines of Γ one meeting Ω and the other meeting $O - \Omega$ is

$$N := \sum_{p \in \mathcal{G}} \mu_p (t^2 + t + 1 - \mu_p)$$

since any two such lines meet in a point of \mathcal{G} .

We determine N in the following. Denote by \mathcal{L} the set of affine lines of Γ not meeting Ω and by \mathcal{M} the set of affine lines of Γ meeting Ω . Let $l \in \mathcal{L}$ and $m \in \mathcal{M}$, i.e. $l^\infty \in O \cap H = O - \Omega$ and $m^\infty \in H - O$. We set

$$\begin{aligned} \mu^-(l) &:= \sum_{p \in l} \mu_p \text{ and} \\ \mu^+(m) &:= \sum_{p \in m \cap \mathcal{G}} \mu_p . \end{aligned}$$

The number $\mu^-(l)$ is the number of affine lines of $\Gamma = \Delta - H$ concurrent with l and meeting O in a point of $\Omega = \Gamma \cap O$. Then the number N of pairs (l, m) of concurrent affine lines with $l \in \mathcal{L}$ and $m \in \mathcal{M}$ is

$$N = \sum_{l \in \mathcal{L}} \mu^-(l) .$$

The following proposition determines the numbers $\mu^-(l)$ and $\mu^+(m)$ showing the numbers are independent from the choice of the particular lines $l \in \mathcal{L}$ and $m \in \mathcal{M}$. Note that the number $\mu^+(m)$ will be used lateron, too.

Proposition 5 For $l \in \mathcal{L}$ and $m \in \mathcal{M}$, it holds $\mu^-(l) = (s-1)(t^2+t)$ and $\mu^+(m) = (s-1)(t^2+t+1) - (t^2+t)$.

Proof. To count the points of Ω collinear with l , we have to subtract the number M of points of $O \cap H$ collinear with l from the number $s(t^2+t)$ of all points of O collinear with l in Γ . Since in Δ , l meets $\Delta_2(\infty)$ in the point $l^\infty = l \cap H$, there is a unique quad δ on ∞ meeting l , namely in l^∞ . For each quad $\omega \neq \delta$ on ∞ , the affine line l is projected by π_ω onto the line $\pi_\omega(l)$ which does not go through ∞ (cf. Proposition 3). Hence $\pi_\omega(l)$ contains exactly one point of $(O \cap H) - \{\infty\}$. For any two quads $\omega, \tau \neq \delta$ on ∞ , the points $\pi_\omega(l) \cap O$ and $\pi_\tau(l) \cap O$ are distinct. Hence there are t^2+t points of $O \cap H$ collinear with l . It follows

$$\sum_{p \in l} \mu_p = s(t^2+t) - (t^2+t) = (s-1)(t^2+t) .$$

Similarly, let γ be the quad on ∞ meeting the line m . As before, for each quad $\sigma \neq \gamma$ on ∞ , $\pi_\sigma(m)$ contains a unique point of $(O \cap H) - \{\infty\}$. Since the affine part of m has only $s-1$ points of \mathcal{G} and there are t^2+t quads on ∞ distinct from γ , it follows

$$\sum_{p \in m \cap \mathcal{G}} \mu_p = (s-1)(t^2+t+1) - (t^2+t) \quad \square$$

In particular, $\mu^-(l)$ and $\mu^+(m)$ do not depend on the lines l and m . Thus $N = \sum_{l \in \mathcal{L}} \mu^-(l) = |\mathcal{L}| \cdot \mu^-(l)$ for some line $l \in \mathcal{L}$. Since \mathcal{L} is the set of affine lines meeting $H \cap O$ and since $|H \cap O| = st(t^2+t+1)$, it follows $|\mathcal{L}| = st^3(t^2+t+1)$. Thus the number N follows:

Corollary 6 $N = st^4(s-1)(t+1)(t^2+t+1)$ \square

To determine $\sum_{p \in \mathcal{G}} \mu_p$, consider the following partition of the point set of $\mathcal{G} = \Gamma - O$. Let κ be a quad on ∞ . Then the affine lines of Γ meeting κ , considered as lines of Δ , partition the point set of Γ . These affine lines fall in two classes K^- and K^+ where K^- is the set of affine lines meeting κ in points of $O - \{\infty\}$, i.e. their affine part does not meet Ω , and K^+ is the set of affine lines meeting κ in points of $(\kappa - \infty^\perp) - O$, i.e. their affine part meets Ω . Since $|\kappa \cap O| = st+1$, it follows $|(\kappa - \infty^\perp) - O| = st(s-1)$. Thus $|K^-| = st^3$ and $|K^+| = st^3(s-1)$. It follows

$$\begin{aligned} \sum_{p \in \mathcal{G}} \mu_p &= |K^-| \mu^- + |K^+| \mu^+ \\ &= st^3(s-1)(t^2+t) + st^3(s-1)((s-1)(t^2+t+1) - (t^2+t)) \\ &= st^3(s-1)^2(t^2+t+1) \end{aligned}$$

With Corollary 6 it follows

$$\begin{aligned}
\sum_{p \in \mathcal{G}} \mu_p^2 &= (t^2 + t + 1) \sum_{p \in \mathcal{G}} \mu_p - N \\
&= (t^2 + t + 1) st^3 (s - 1)^2 (t^2 + t + 1) - st^4 (s - 1) (t + 1) (t^2 + t + 1) \\
&= st^3 (t^2 + t + 1) (s - 1) ((t^2 + t + 1)(s - 1) - (t^2 + t))
\end{aligned}$$

We are now in the position to conclude the proof by Cauchy's inequality

$$|\mathcal{G}| \sum_{p \in \mathcal{G}} \mu_p^2 \geq \left(\sum_{p \in \mathcal{G}} \mu_p \right)^2 .$$

Indeed, it holds

$$|\mathcal{G}| \sum_{p \in \mathcal{G}} \mu_p^2 = s^2 t^6 (s^2 - s + 1) (t^2 + t + 1) (s - 1) ((t^2 + t + 1)(s - 1) - (t^2 + t))$$

leading with $\sum_{p \in \mathcal{G}} \mu_p = st^3 (s - 1)^2 (t^2 + t + 1)$ to the inequality

$$(s^2 - s + 1) ((t^2 + t + 1)(s - 1) - (t^2 + t)) \geq (s - 1)^3 (t^2 + t + 1) .$$

This is equivalent to

$$s^2 - s - t^2 - t \geq 0 .$$

Note that the counting arguments do not depend on the particular polar space under consideration. In particular, they are independent from the parity of s or t .

Since the existence of an ovoid in a classical generalized quadrangle forces $s \geq t$, the inequality leads to a contradiction only if $s = t$. From the dual polar spaces admitting ovoids in quads, only $\Delta \cong DW(6, q)$ has orders $s = t = q$ leading to the contradiction $2q \leq 0$. Hence the dual polar space $DW(6, q)$ does not admit any ovoid whereas we cannot deduce anything about the two other classical dual polar spaces $DO^-(8, q)$ admitting ovoids of quads and $DH(7, q^2)$ possibly admitting ovoids of quads.

References

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